Working Paper
Series

FINRISK
National Centre of Competence in Research
Financial Valuation and Risk Management

National Centre of Competence in Research
Financial Valuation and Risk Management

Working Paper No. 316

Market Selection in an Evolutionary Market with Creation and Disappearance of Assets

Urs Schweri

First version: July 2006
Current version: November 2011

This research has been carried out within the NCCR FINRISK project on
"Evolution and Foundations of Financial Markets"

Market Selection in an Evolutionary Market with Creation and Disappearance of Assets*

Urs Schweri†

November 14, 2011

Abstract

Identifying investment strategies that will survive in the long run is a main endeavor in the field of evolutionary finance. The evolutionary perspective on the financial market considers rather long time horizons, making the creation and disappearance of firms a highly relevant factor in determining such strategies. However, this factor has not been examined in existing research. This paper seeks to fill the gap in the literature by simulating dividends and investment strategies on the basis of initial public offerings (IPOs) and defaults. This paper simulates the evolution of the wealth shares of various investment strategies in a setup wherein dividends are nonstationary. The results show that a modified version of the generalized Kelly rule dominates competing investment strategies in terms of final wealth. This finding agrees with the existing literature, which suggests that the generalized Kelly rule has good chances of surviving or even taking over the entire market in different setups. However, the creation and dissolution of a firm can only be observed once in the life of a company; therefore, using only a long time series of one company alone is not the most optimal method of estimating the probability that a firm will default. Instead, the dividend process must be understood by examining similar companies. This completely alters the implementation of the generalized

*I would like to thank Thorsten Hens, Klaus Reiner Schenk-Hoppé, and Andreas Tupak for their valuable input. Financial support by the National Center of Competence in Research "Financial Valuation and Risk Management" is gratefully acknowledged. The National Centers in Research are managed by the Swiss National Science Foundation on behalf of federal authorities.

†Urs Schweri, University of Zurich, Department of Banking and Finance, Plattenstrasse 32, 8032 Zürich, www.bf.uzh.ch, urs.schweri@bf.uzh.ch, +41 (0)44 634 48 22

Kelly rule compared with the way it is applied in the existing evolutionary finance literature, even when the dividend processes of the companies involved are independent of each other.

1 Introduction

Financial analysis is based on the rationale that companies with similar characteristics exhibit a comparable firm value. One possible explanation for this may be that events that can only be observed once in the entire life of a company have a tremendous impact on the future of that company. Examples of such events are the first blockbuster product of a biotechnology company, the development of the iPhone by Apple Inc., or the default of a company. Moreover, such events can only be studied by examining similar companies. Further, this paper seeks to demonstrate that, in a market with several investment strategies, investors who incorporate cross-sectional information (i.e., information from other firms) in their investment decisions perform better than those who do not. In other words, the share of the total wealth accumulated using the strategies of the former increases more quickly than that accrued by those of the latter. Accordingly, strategies that perform poorly are marginalized in the long run. Briefly, this paper will show that the market selects investors who use cross-sectional information and that other strategies disappear in the long term.

The idea that the market selects investors who use all available information and who act rationally was initially proposed by Friedman (1953) and Fama (1965). According to these researchers, irrational investors earn lower returns and disappear in the long run. However, Long et al. (1990) used a partial equilibrium model to show that the effects of decisions made by irrational investors on stock prices cannot always be corrected by rational investors because the latter are risk averse. In addition, Blume and Easley (1992) proved that a rational investor who does not maximize a logarithmic utility function can be driven out of a complete market by some irrational investors, assuming that every investor has the same savings rate. For exogenous asset prices, Kelly (1956) developed a theory of maximizing expected returns on long-term (financial) investments. To do so, the investor has to bet his beliefs. Maximizing the growth rate as the Kelly rule does is equivalent to maximizing a logarithmic utility function. Therefore, a slightly irrational investor who almost maximizes a logarithmic utility function can push a rational investor who maximizes a nonlogarithmic utility function out of the market on the basis of the higher growth rate of his wealth as in Blume and Easley (1992).

Samuelson (1979) argued that individuals should maximize their utility (and therefore their happiness), regardless of whether they survive in the market. However, this paper focuses on identifying the strategies that survive a market selection process, rather than on making people happy. Because of the exogenously given savings rate, the asset allocations in Blume and Easley (1992) are neither Pareto optimal nor do they have a general equilibrium model. Sandroni (2000) and Blume and Easley (2006) investigated market selection in a general equilibrium setting with complete markets, and found that rational investors survive when all investors have the same discount rate, but the same does not apply in incomplete markets. These are addressed in detail by Evstigneev et al. (2006), and Evstigneev et al. (2008) showed that if all strategies and dividends possess Markov properties or that dividends are independent and identically distributed (i.i.d.), the generalized Kelly rule will drive all strategies that depend only on the actual state of the world out of the market, given that the initial wealth of the competing strategies is small enough (this property is called local evolutionary stability). This is further generalized by Amir et al. (2009b); they found that the generalized Kelly rule is asymptotically unique among all survival strategies that depend only on the history of states. This implies that the Kelly rule has almost surely a strictly positive wealth share that is independent of the strategy of the other investors. However, asset prices depend not only on the past states but also on the strategies of the other investors. Therefore, these results are for many relevant strategies as for example momentum strategies not applicable. However, simulation results from Tupak (2009) indicate that the generalized Kelly rule will dominate, given that the true parameters of the dividend process are known.[1]

What types of investment strategies survive if dividends are nonstationary and assets can be created and dissolved? Econometricians have long debated whether dividends contain a unit root or follow a stationary process, and the discussion is still not completely resolved.[2] Summarizing this debate, the test statistics of the unit root tests suggest that the hypothesis of a unit root in dividends is more difficult to maintain than the hypothesis of a unit root in stock prices. Further, Harris and Tzavalis (2004) have rejected the unit root hypothesis for dividends, and DeJong and Whiteman (1991) have also found it implausible. Therefore, this paper will concentrate on the implications of the second market feature, which states that assets can be created and destroyed. This feature automatically generates a nonstationary dividend process because many companies, including large ones that are very stable in the short term, did not exist, say, 200 years ago. To determine strategies

[1]The generalized Kelly rule also applies to one-period assets with an arbitrary dividend process, see Evstigneev et al. (2002), Hens and Schenk-Hoppé (2005), and Amir et al. (2005).

[2]For arguments surrounding the existence of a unit root in dividends, see Shiller (1981), Kleidon (1986), Campbell and Shiller (1987), Campbell and Shiller (1988), DeJong and Whiteman (1991), and Harris and Tzavalis (2004).

that will survive in the long run, it is therefore important to consider the fact that companies can disappear and new companies will enter the market.

It is often assumed that dividends are driven by one and the same process over the entire life of a company. This is a very simplistic assumption: for instance, why should a small startup have the same risk and expected returns as a large concern? Mueller (1972) suggested a firm life-cycle: small firms are more profitable and face greater risks than large ones, but the large firms pay greater dividends. This life-cycle is driven by the idea that small firms tend to be more innovative but have difficulties in accessing the credit market and are therefore unable to pay dividends. Hall (1987) and Evans (1987a,b) found support for this theory in their work on US manufacturing firms, which led them to conclude that small firms grow more quickly and are riskier than larger firms. Similarly, Dhawan (2001) discovered that small US manufacturers are more productive and riskier than large ones. Fama and French (2001), Grullon et al. (2002), DeAngelo and DeAngelo (2006), and DeAngelo et al. (2006) provided empirical evidence for the life-cycle hypothesis of dividend policy, which holds that large firms pay more dividends than small, growing companies. In addition to the possibility of default and the constructible and destructible nature of firms, this paper will consider the fact that small firms with small dividend payments may become large firms with high dividend payments.

The main aim of this paper is to find a surviving strategy for nonstationary dividends modeled on the creation and destruction of companies. Since the generalized Kelly rule is not feasible in this setup, the adopted strategy ensures that funds that are invested into a company are proportional to the expected net present value (NPV) of that companys dividends. This paper makes several observations. First, the NPV-strategy is able to dominate the markets in simulations; this indicates that the results from the infinitely lived assets seem to generalize to this setup. Second, many observations are required if the parameters of the process and the portfolio weights have to be estimated from past observations of the dividends. If past observations of dividends are lacking, a generalized Kelly rule with estimated parameters can be driven out of the market using simple strategies. This confirms the theoretical findings pertaining to infinitely lived assets obtained by Amir et al. (2009a) and the simulation findings for stationary dividends obtained by Tupak (2009). Third, if a wrong dividend process is assumed, the optimal Kelly rule can produce worse results than a naïve strategy that invests the same amount in each asset. This is shown in a case were the investor assumed the dividends to be i.i.d.,[3] while in reality, the dividends followed a nonstationary process.

Section 2 provides empirical evidence that the creation and disappearance of companies is an important factor in the dividend process and discusses further stylized facts. Section 3 presents a simple and minimalist dividend model that conforms to the literature discussed and the empirical evidence presented herein. Section 4 describes the market selection model in which the investment strategies detailed in Section 5 will compete. Section 6 simulates some models and Section 7 summarizes the papers findings.

2 Empirical evidence on the birth, death, and dividends of companies

The present section motivates the assumptions for the dividend process described in the next section. This section mainly shows that, over the last 40 years, many new firms have been founded and are default, a fact that is often neglected in evolutionary finance. Furthermore, dividend payments are largely issued from a small number of companies, and the percentages of dividends paid by different sectors change over time. All of this demonstrates that dividends are highly nonstationary.

To illustrate these points, a sample of 25,272 active and inactive North American companies that are listed on the New York Stock Exchange (NYSE), the American Stock Exchange (AMEX), or the NASDAQ Stock Market is used. The data have been obtained from the Center for Research in Security Prices (CRSP), and they are for the period 1973 through 2009. The starting date of 1973 was chosen because that is the earliest year from which AMEX and NASDAQ data are available. A company is considered active as long as it is listed on a certain stock exchange. Within this period, the number of active companies varies from 5,267 to 9,843 per year; the average is 7,138 companies per year. The large difference between the number of active companies per year and the number of active and inactive companies indicates that many firms were newly founded and that a large number of companies disappeared.

A company is active for an average time period of 10.51 years, and the median is lower: eight years. Figure 1 shows the number of companies that

[3]This assumption was used in the simulations of Hens et al. (2002), Hens and Schenk-Hoppé (2004), and Tupak (2009).

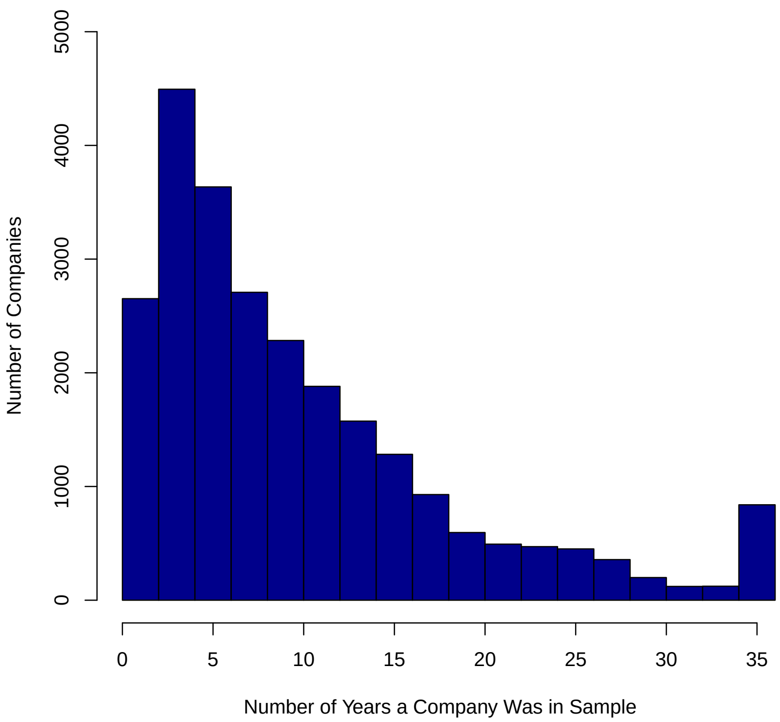


Figure 1: Number of years a company was in the sample (CRSP data from 1973 to 2009).

remain in the sample for any given number of years. Approximately a quarter of all firms were in the sample for less than four years, thus making it extremely difficult to determine the value of an asset on the basis of its past dividends. Therefore, cross-sectional information may be helpful in ascertaining the value and risk and therefore also the optimal amounts of investment in such assets.

The largest part of the variation in the number of active firms can be explained by mergers and acquisitions (see Table 1). Of the 18,837 delistings reported by the CRSP from 1973 to 2009, a total of 9,782 are attributable to this source. The next most important reason for deletion is being dropped from the stock exchange. The number of dropped companies is much higher than the number of liquidated companies, and this indicates that the big stock exchanges delist companies with financial problems before the worst happens. Table 2 provides detailed reasons for dropping the companies: 1,282 companies were delisted because of insufficient capital, 930 because their price was too low, 647 because of insolvency, and 982 because they did not pay exchange fees. Therefore, the proportion of defaulting companies accounts for at least 12.5% of all companies, based on a time period of 36 years. The number of new companies is also significant: 19,318 such companies emerged during the period under study, working out to an average of 536.6 per year. These figures plainly demonstrate that long-run investment strategies should not neglect the fact that firms have finite lives.

Neither the numbers of delisted companies nor the reasons these companies were delisted are constant over time (see Table 1). Typically, everything happens in waves. For example, many new companies were founded between 1991 and 1997, and a merger wave occurred from 1996 to 2001. Between 1998 and 2004, the number of companies fell and the number of liquidations increased tremendously, and this phenomenon was repeated in 2008 and 2009. Due to this cyclical pattern, the number of companies also moves in waves. These findings correspond with the initial public offering (IPO) waves discovered by Ibbotson and Jaffe (1975), the procyclical behavior of IPOs noted by Pástor and Veronesi (2005), and the countercyclical behavior of default probabilities observed in Vassalou and Xing (2004), Chava and Jarrow (2004), and Chen (2010).

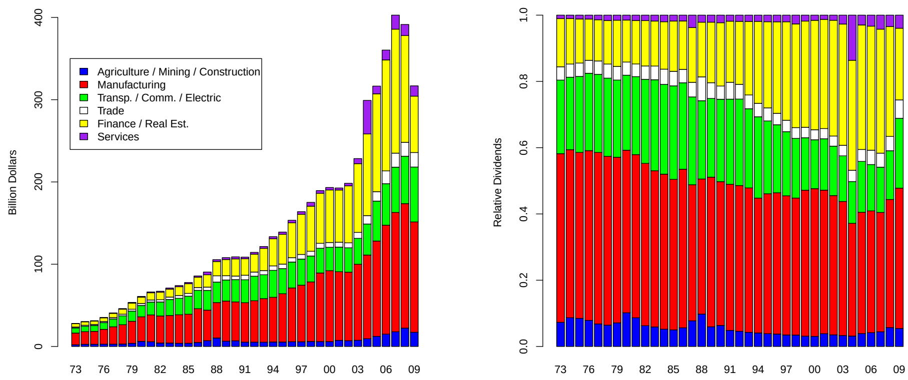
To aggregate dividends data from the CRSP, the number of outstanding shares on the day before the ex-distribution date must be multiplied by the dividend per share and then aggregated over one calendar year. For these calculations, only cash dividends were taken into account (i.e., subscription rights etc. were excluded from the estimations). Figure 2 aggregates the divi-

Year	Active	Mergers	Liquidation	Dropped
1973	5,954	103	5	354
1974	5,558	99	11	168
1975	5,398	83	10	80
1976	5,415	101	17	53
1977	5,390	155	18	61
1978	5,344	191	11	70
1979	5,267	209	16	51
1980	5,431	164	24	83
1981	5,794	155	15	88
1982	6,008	172	21	159
1983	6,606	173	11	164
1984	6,862	210	13	229
1985	6,982	248	16	325
1986	7,375	225	26	319
1987	7,642	185	5	225
1988	7,655	352	12	312
1989	7,390	273	13	318
1990	7,218	197	9	339
1991	7,251	119	12	367
1992	7,538	135	9	380
1993	8,108	177	3	170
1994	8,676	279	3	198
1995	9,055	358	9	233
1996	9,608	437	10	174
1997	9,843	511	7	255
1998	9,695	603	5	422
1999	9,374	613	11	394
2000	9,055	631	11	326
2001	8,337	463	7	473
2002	7,653	259	17	390
2003	7,228	257	11	297
2004	7,064	265	17	147
2005	7,043	268	7	162
2006	6,971	312	7	123
2007	7,000	392	8	184
2008	6,563	250	22	230
2009	6,237	158	36	267
Total		**9,782**	**465**	**8,590**

Table 1: Active companies (at the end of the year) and reasons for delisting (CRSP data from 1973 to 2009)

Reason for Dropping	Firms
Issue stopped trading on current exchange–reason unavailable	965
Issue transferred from current exchange to Mutual Funds	18
Issue transferred from current exchange to Boston Exchange	33
Issue transferred from current exchange to Midwest Exchange	2
Issue transferred from current exchange to Pacific Stock Exchange	17
Issue transferred from current exchange to Philadelphia Stock Exchange	3
Issue transferred from current exchange to Toronto Stock Exchange	3
Issue began trading over the counter	375
Delisted by current exchange–insufficient number of market makers	464
Delisted by current exchange–insufficient number of shareholders	170
Delisted by current exchange–price fell below acceptable level	930
Delisted by current exchange–insufficient capital, surplus, and/or equity	1,282
Delisted by current exchange–insufficient (or noncompliance with rules of) float or assets	707
Delisted by current exchange–company request (no reason given)	512
Delisted by current exchange–company request (deregistration owing to going private)	81
Delisted by current exchange–bankruptcy (declared insolvent)	647
Delisted by current exchange–company request (offer rescinded and issue withdrawn by underwriter)	15
Delisted by current exchange–delinquent in filing and non-payment of fees	982
Delisted by current exchange–failure to register under Section 12G of the Securities Exchange Act	112
Delisted by current exchange–failure to meet exception or equity requirements	167
Delisted by current exchange–denied temporary exception requirement	10
Delisted by current exchange–does not meet exchanges financial guidelines for continued listing	867
Delisted by current exchange–protection of investors and the public interest	137
Delisted by current exchange–corporate governance violation	13
Conversion of a closed-end investment company to an open-end investment company	47
Delisted by current exchange–required by the Securities Exchange Commission (SEC)	31

Table 2: Reasons that companies were dropped from their exchange (CRSP data from 1973 to 2009)



(a) Dividends by industrial sector

(b) Industrial sector dividends relative to total dividends

Figure 2: Dividends by Standard Industrial Classification (SIC) sector and SIC dividends relative to total dividends, that is, the dividends of one sector divided by total dividends (CRSP data from 1973 to 2009)

dends by sector, and Figure 2(b) shows the relative weight of dividends being paid by different sectors. Until 2007, the dividends paid by the financial sector increased at a faster rate than those paid by the manufacturing industry and the transport and telecommunication sector. After the financial crisis in 2008, the dividends of the financial sector reduced drastically. This reveals persistent shifts in the relative weight of dividends being paid by different sectors. Such shifts are quite natural; the railroad and textile industries, for instance, were much more important one hundred years ago than they are today. Long-term shifts are incompatible with the assumption of i.i.d. dividend shares of the different sectors (or companies), and this assumption has often been made by parts of the evolutionary finance literature.

No dividends were paid in 57.2% of all company years, which are defined as the years during which a company is active. In 5.5% of all company years, dividends increased from zero to a positive amount and in 5.3% of the years, dividends fell to zero. This indicates that large variations in dividends are a very characteristic feature of dividend time series. Furthermore, the top 5% of dividend payers[4] distributed, on average, 78% of all dividends. This percentage varied between 71% and 85%, reaching its lowest in 1979 and peaking in 2001. These figures show that dividend payment is enormously

[4]The top 5% of dividend payers constitute 5% of the companies paying the highest total dividends.

concentrated among a few firms and that this concentration trended upward over time.

This section has noted several patterns relating to company numbers and dividend payment. To reflect these patterns, a long-run dividend model should exhibit the following features: a dynamic number of companies, wave-like changes in the number of companies, some large jumps in dividends, concentration of dividend payments among a small fraction of firms, and the capacity to accommodate shifts of dividends between different sectors.

3 Dividend model

This section will present a dividend model that is based on the stylized facts established in the previous section. Within this model, firms can be born and default, and a few large corporations issue a large percentage of dividend payments. In other words, small, young firms (startups) pay only a small dividend and are at high risk of defaulting, but have opportunities to grow into concerns, which pay large dividends. To provide more detail, this model assumes an economy that consists of three types of companies: IPOs, startups, and concerns. IPOs are firms that have newly entered the market. In the entering period, investors pay a certain amount for an IPO and do not receive a dividend, and in the second period, the IPO automatically becomes a startup. Startups pay low dividends and may grow into concerns, which are mature firms that spend large sums on dividends but cannot grow further. These characteristics reflect the empirical evidence produced by Hall (1987), Evans (1987a,b), and Dhawan (2001), which demonstrates that younger firms have higher growth rates than older firms. Both types of companies can default and a company may change types in any period (a dead company being one such type). That is to say, a startup can default, remain a startup, or become a concern, and a concern can either default or remain a concern. A company that has defaulted is dead forever. The transition probabilities in Figure 3 are given as follows: p_{SD} is the probability that a startup will default during a period and p_{SS} is the probability that the startup will remain a startup. If the startup survives, then p_{SD} is the probability that it will default during the next period. The probability that a startup becomes a concern is then given by $p_{SC} = 1 - p_{SS} - p_{SD}$. A concern may remain a concern or default, and the probabilities for these events are p_{CC} and $p_{CD} = 1 - p_{CC}$.

Type changes are independent between companies and over time. If p_{SD} and p_{CD} are strictly positive, every company will default at one point in

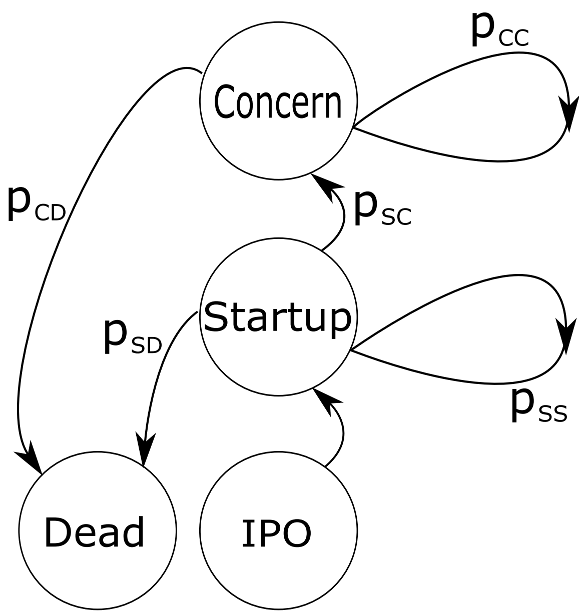


Figure 3: Development of a company over time. An IPO becomes a startup, a startup can become a concern or disappear after several periods, and a concern will exist for several periods and then disappear. The probability of each event is presented beside the arrows.

time, that is, if $t \to \infty$ the probability that a company is defaulted converges to 1. To guarantee that some companies will always exist, the number of IPOs in every period, n_{new}, exceeds zero. The number of startups, concerns, and IPOs are represented by n_t^S, n_t^C, and n_{new}, respectively. The long-run averages of the number of startups and concerns can be calculated as follows:

$$\mathbb{E}\left(n^S\right) = \frac{n_{new}}{1 - p_{SS}} \quad \text{and} \quad \mathbb{E}\left(n^C\right) = \frac{p_{SC} \cdot n_{new}}{(1 - p_{SS})(1 - p_{SC})}. \tag{1}$$

Every year, every startup pays a fixed strictly positive dividend D^S, and each concern pays a fixed dividend $D^C > D^S$. The fixed dividend D^S remains constant from the foundation of a startup to the point where it either becomes a concern or dies. Should it become a concern, the dividend of the new concern experiences a huge upward jump; should it die, however, D^S permanently falls to zero. A concern pays D^C every year until it is dissolved. Therefore, the sum of all dividends paid by startups in a single period is the result of $n_t^S D^S$ and the total of all dividends paid by concerns during the same period is $n_t^C D^C$. Since IPOs do not pay dividends, it follows that the total dividend paid out in period t is the sum of the dividends paid by the startups and the concerns. The general structure of this dividend model is

Startup		Concern		Miscellaneous	
D^S	1	D^C	40.1	n_{new}	1
p_{SD}	2.3%	p_{CD}	0.6%	λ_0	5%
p_{SS}	97.0%	p_{CC}	99.4%		
p_{SC}	0.7%				

Table 3: Simulation parameters calibrated on CRSP data

not completely new. Hurley and Johnson (1994) used a similar trinomial model to price individual stocks.

This model incorporates most of the features mentioned at the end of Section 2. The number of companies is dynamic and undergoes wave-like changes. Dividend payments can be parameterized so that they are largely paid by concerns and only fractionally disbursed by startups and are therefore concentrated among the concerns. To ensure that the simulation problem in Section 6 is tractable, the number of IPOs in every period is set to one. Simulations show that, with this assumption, the number of startups and concerns is changing drastically over time. Adding waves in the number of IPOs, as observed in the data of Section 2, would strengthen this effect further. Mergers are not included in the model because they do not matter, assuming that the dividends and portfolio weights of the new firm are the sum of the merging companies. Owing to the fact that new companies enter the market at all the times and existing companies disappear in the long run, changes in the dividends between several sectors can be explained by the model: in a certain time frame, mainly textile firms could enter the market, while in another period, only IT firms, and so on. That is, the sector of firms entering the market changes over time. After a certain period, the IPOs become concerns and pay considerable dividends, thereby leading to an increase in the importance of the sector. If no new firm of a certain sector enters the market, the sector disappears in the long run. As a whole, the model is very simple, but it includes many elements that are important in the long-run dividend process.

This paper aims to simulate the wealth shares of different investment strategies. For this, dividends must also be simulated. Table 3 shows the parameters of the dividend process. These are calibrated with CRSP data for the years 1973 to 2009 in order to correlate the dividend process with the stylized facts underlying the dividend model. Table 2 does not confirm whether companies that were dropped from their exchanges were delisted owing to financial problems. Because of that uncertainty, the decision regarding which type a company belongs to is based on market capitalization:

a company that has belonged to the top decile of all companies for at least two years is classified as a concern from that point until it defaults. Companies that neither qualified as concerns nor belonged to the lowest decile for at least two years in a row are classified as startups. A startup defaults if its value remains in the lowest decile for the rest of its life, whereas a concern defaults if its value remains below the largest 30% of all active companies for the rest of its life. The default threshold for concerns may seem high, but the market value of a concern defined as dead is approximately five times lower than the market value at which a startup becomes a concern. In other words, this threshold ensures that a concern must have suffered substantial losses before it defaults. The dividends of a startup, D^S, is normalized to one. The concern dividends, D^C, are determined in two steps. First, the quotient obtained by dividing every year the average of concern dividends through the average of startup dividends. This results in the dividends of the concerns in every year (D^S is normed to 1). Given that, the dividend of the concern is the average over all the years from which this quotient is derived. This dividend process is unrealistic for two reasons: first, it does not consider mergers, and second, it assigns equal dividend amounts to all company types. However, these simplifications allow us to observe the effects of the creation, growth, and default of firms on the wealth of investment strategies, which is the main purpose of this model.

4 Market selection model

As stated in the previous section, the dividend model is based on exogenously given dividends. In the next step of the dividend process, the companies generating these dividends are traded in a market and their shares may be purchased by several investors. This section will describe how the wealth of differing investment strategies and with that the asset returns evolve over time.

The state of nature in t is $\omega(t)$ and is described by the dividend payment of all companies at t. Therefore, the history of states equals $\omega^t = (\omega(0), \ldots, \omega(t))$. Given this, the percentage of wealth consumed by investment strategy i, at t in an economy with I investment strategies is defined as $\lambda^i_{0,t}(\omega^t)$, where $i \in \{1, \ldots, I\}$. This percentage is assumed to be constant over time and identical for all strategies because this paper focuses on comparing the performance of investment strategies, rather than analyzing the influence of the savings rate. Further, this assumption allows us to simplify $\lambda^i_{0,t}(\omega^t)$ to λ_0, which is important because it eliminates the possibility that an irrational strategy will survive by having a higher savings rate than rational strategies. In such a case, the rational strategy has a higher return, but the irrational strategy achieves a higher growth rate through a higher savings ratio and thereby marginalizing the rational strategy in the long run (see, e.g., Blume and Easley (1992)).

The percentage of wealth invested by shareholder i in company k at time t is represented by $\lambda_{k,t}^{i}\left(\omega^{t}\right)$. Nonexistent companies must have portfolio weights of zero; therefore, $\lambda_{k,t}^{i}\left(\omega^{t}\right)=0$ for all companies that do not exist at t. Every strategy can invest in any existing company, but short selling is not allowed; that is, $0 \leq \lambda_{k,t}^{i}\left(\omega^{t}\right) \leq 1$. This budget constraint implies that $\sum_{k} \lambda_{k,t}^{i}\left(\omega^{t}\right)=1$. Note that the sum over k can be interpreted as the sum over all past, actual, and future companies. However, it is not possible to invest in nonexistent companies, so this equates to adding up only the investments made in companies existing in period t. The wealth of investor i in t is w_{t}^{i} and the price of asset k in t is $q_{k,t}$. Therefore, the number of shares held by investor i in company k at time t is

$$\theta_{k,t}^{i}=\begin{cases}\frac{\lambda_{k,t}^{i}\left(\omega^{t}\right) w_{t}^{i}}{q_{k,t}} & \text{if company } k \text{ exists in } t \\ 0 & \text{otherwise.}\end{cases} \tag{2}$$

If the number of stocks issued by a company is normalized to one and if all stocks need to be held by someone, the price of one stock in company k at t, or the market capitalization of that company, can be given as follows:

$$q_{k,t}=\sum_{i=1}^{I} \lambda_{k,t}^{i}\left(\omega^{t}\right) w_{t}^{i}=: \boldsymbol{\lambda_{k,t}}\left(\omega^{t}\right) \boldsymbol{w}_{t} \tag{3}$$

where $\boldsymbol{w}_{t}$ is a column vector including the wealth of all investors in period t and $\boldsymbol{\lambda_{k,t}}\left(\omega^{t}\right)$ is a row vector with the portfolio weights of all investors in asset k during period t. The asset prices can be written in matrix notation as follows:

$$\boldsymbol{q}_{t}=\boldsymbol{\Lambda}_{t}\left(\omega^{t}\right) \boldsymbol{w}_{t}, \tag{4}$$

where $\boldsymbol{q}_{t}$ is a price vector including all companies and $\boldsymbol{\Lambda}_{t}\left(\omega^{t}\right)$ is a matrix of all portfolio weights of period t. The number of rows represents the number of assets, and the number of columns represents the number of investors. Finally, the vector $\boldsymbol{w}_{t}$ includes the wealth of every investor in t. The wealth of an investor in $t+1$ is equal to the value of his portfolio plus the dividend payment:

$$w_{t+1}^{i}=\sum_{k}\left(D_{t+1}^{k}\left(\omega^{t+1}\right)+q_{k,t+1}\right) \theta_{k,t}^{i}. \tag{5}$$

The dividends of asset k at $t+1$ are represented by $D_{t+1}^{k}\left(\omega^{t+1}\right)$. Note that $q_{k,t+1}$ and $\theta_{k,t}^{i}$ depend on $\boldsymbol{w}_t$ and $\boldsymbol{\lambda_{k,t}}\left(\omega^{t}\right)$. The next step is to express the wealth dynamic in terms of exogenously given variables as the dividends and the strategy that depends only on the state of the world, ω_t. The preceding equation can be expressed in matrix notation as follows:

$$\boldsymbol{w}_{t+1} = \sum_{k}\left(D_{t+1}^{k}\left(\omega^{t+1}\right) + q_{k,t+1}\right)\boldsymbol{\theta}_{k,t}, \tag{6}$$

$$= \boldsymbol{\Theta}_t \boldsymbol{D}_{t+1}\left(\omega^{t+1}\right) + \boldsymbol{\Theta}_t \boldsymbol{q}_{t+1}, \tag{7}$$

$$= \boldsymbol{\Theta}_t \boldsymbol{D}_{t+1}\left(\omega^{t+1}\right) + \boldsymbol{\Theta}_t \boldsymbol{\Lambda}_{t+1}\left(\omega^{t+1}\right)\boldsymbol{w}_{t+1}, \tag{8}$$

where the last equation follows from equation (4). The number of shares held by all investors in all companies, $\boldsymbol{\theta}_{k,t}$, is a column vector with the length of the number of investors and $\boldsymbol{\Theta}_t$ combines the vectors $\boldsymbol{\theta}_{k,t}$ into a matrix in which the number of investors is designated by the number of rows and the number of assets by the number of columns. Furthermore, $\boldsymbol{D}_{t+1}\left(\omega^{t+1}\right)$ is the vector denoting the dividends of all assets in $t+1$, given the state ω^{t+1}. Writing $\boldsymbol{w}_{t+1}$ on one side of the equation results in

$$\left(\mathbf{I} - \boldsymbol{\Theta}_{\mathbf{t}}\boldsymbol{\Lambda}_{\mathbf{t+1}}\left(\omega^{\mathbf{t+1}}\right)\right)\boldsymbol{w}_{t+1} = \boldsymbol{\Theta}_t \boldsymbol{D}_{t+1}\left(\omega^{t+1}\right). \tag{9}$$

The evolution of wealth is therefore

$$\boldsymbol{w}_{t+1} = \left(\mathbf{I} - \boldsymbol{\Theta}_{\mathbf{t}}\boldsymbol{\Lambda}_{\mathbf{t+1}}\left(\omega^{\mathbf{t+1}}\right)\right)^{-1}\boldsymbol{\Theta}_t \boldsymbol{D}_{t+1}\left(\omega^{t+1}\right). \tag{10}$$

The next step is to check whether the wealth of all investors in $t+1$, $\boldsymbol{w}_{t+1}$, and the stock prices during the same period, $\boldsymbol{q}_{t+1}$, are always nonnegative. This is important because negative asset prices make no economic sense and negative wealth presents the problem of whether the investor will be able to pay back his or her debts. The system can be considered well-defined if the wealth of all investors and prices of all assets are nonnegative during all t. This requires three assumptions:

Assumption 1. *Consumption takes place and does not violate the rule that* $0 < \lambda_{0,t}^{i}\left(\omega^{t}\right) < 1$ *for all* i, t, *and* ω^{t}.

Assumption 2. *At least one completely diversified portfolio rule is in force: an* i *exists such that* $\lambda_{k,t}^{i}\left(\omega^{t}\right) > 0$ *for all existing* k, t, *and* ω^{t}.

Assumption 3. *If a company,* k, *is dead or not yet founded at* t, *then nobody invests in it (i.e.,* $\lambda_{k,t}^{i}\left(\omega^{t}\right) = 0$*).*

Proposition 1. *Suppose that* $w_0 > 0$ *and assumptions 1 to 3 are satisfied. Then, the evolution of wealth* (10) *is well defined in all* $t < \infty$.

Mainly, the proposition holds because this setup is constructed so that both $\mathbf{\Theta}_t$ and $\mathbf{\Lambda}_t(\omega^t)$ contain many zeros; consequently, the step from t to $t+1$ is only influenced by companies existing in both periods. Considering this, the proof for Proposition 1 is analogous to that supplied by Evstigneev et al. (2006). Since the chief effect of Proposition 1 is to enable proper model simulation, the restriction to a finite number of time periods is not problematic.

The model may appear very similar to that of Evstigneev et al. (2006) or Amir et al. (2009a), but it is not possible to show that the generalized Kelly rule is locally evolutionary stable[5] or is a (unique) surviving strategy. A main prerequisite of their result is that consumption is a constant share of total wealth. In the present setup, shareholders pay a certain amount of money to establish a newly founded IPO, and this amount depends on the investment strategies of the investors and is therefore typically not constant over time. This fraction of investor wealth leaves the economy and is hence also a form of consumption, but because it is not constant over time, it is not possible to confirm the existence of principles such as local evolutionary stability or a unique surviving strategy. Results on these subjects are therefore provided by simulations. However, before this, the strategies to be considered for simulations must be defined.

5 Strategies

This paper has thus far delineated a dividend model and an evolutionary market selection model and will now proceed to discuss investor strategies, which must be known in order to simulate the entire market. Which strategies should compete in this model? A good starting point may be a generalized version of the Kelly rule:

$$\lambda^*_{k,t} = \frac{\lambda_0}{1-\lambda_0} \sum_{m=1}^{\infty} (1-\lambda_0)^m \, \mathbb{E}\left(d_{k,t+m}(\omega^{t+m}) \mid \omega^t\right),$$

where $d_{k,t}$ are the relative dividends of asset k at t (i.e., $d_{k,t} = \frac{D^k_t}{\sum_i D^i_t}$). The strategy λ^* has a probability of one of resulting in a positive wealth share when applied to both short (one-period) and infinitely long-lived assets see Amir et al. (2009a,b). However, neither this nor other results from literature dealing with local and evolutionary stability apply to the strategy λ^* in a

[5] A strategy that drives every other strategy out of the market if the initial wealth of the other strategy is small enough is considered locally evolutionary stable.

market wherein new companies can be established or firms can default. Furthermore, the strategy is not directly applicable in the setup of this paper. This is because new companies will enter the market in future periods and will, therefore, pay positive relative dividends. These assets are not included in the calculation of λ^*, which must therefore not sum up to one. This issue can be circumvented by including only those companies that existed during t in the calculation of the relative dividends in the formula for $\lambda^*_{k,t}$. From a practical point of view, the lack of a closed-form solution for calculating λ^* is a more problematic issue. The strategy could be estimated via simulation, but doing so over an infinite time horizon would be time consuming and/or imprecise. Moreover, the evolutionary setup requires that this calculation be performed thousands of times, which was not practicable. Portfolio weights proportional to the NPV of the companies dividends provided a close substitute. The NPV of asset k with discount factor $1 - \lambda_0$ is defined by

$$\mathrm{NPV}_k = \sum_{m=1}^{\infty} (1 - \lambda_0)^m \, \mathbb{E}\left(D_{k,t+m}(\omega^{t+m}) \mid \omega^t\right).$$

The NPVs of the dividends of the different types of companies are as follows:

$$\begin{aligned}
\mathrm{NPV}_{concern} &= \frac{(1-\lambda_0)\, p_{CC} D^C}{1 - (1-\lambda_0)\, p_{CC}} \\
\mathrm{NPV}_{startup} &= (1-\lambda_0)\, \frac{p_{SS} D^S + p_{SC}\left(D^C + \mathrm{NPV}_{concern}\right)}{1 - (1-\lambda_0)\, p_{SS}} \\
\mathrm{NPV}_{IPO} &= (1-\lambda_0)\left(\mathrm{NPV}_{startup} + D^S\right).
\end{aligned}$$

If $NPV_{k,t}$ is defined as the NPV of asset k in period t, then the strategy based on the relative NPVs is as follows:

$$\lambda^1_{k,t}\left(\omega^t\right) = \frac{\mathrm{NPV}_{k,t}}{\sum_j \mathrm{NPV}_{j,t}}.$$

To find out whether the NPV strategy approximates the generalized Kelly rule, the portfolio weights of both strategies were calculated for several parameterizations and then compared. To determine the portfolio weights of the Kelly rule, the relative dividends of the companies were simulated 1,000 periods ahead, and the Kelly strategy was calculated using these dividends. This process was repeated 10,000 times. The average of these 10,000 realizations gives λ^*. This result not only shows that the formula for the NPV is similar to the formula for λ^*. In fact, the NPV strategy and the generalized Kelly rule are equivalent if total dividends in the economy are constant over time. The rest of the paper mainly uses one standard parameterization, which can be found in Table 3. The parameters of the standard parameterization are varied in Table 4, and the difference between the allocation of the Kelly rule, λ^*, and the NPV-strategy, λ_1 of these variations are presented in Panels A to D.[6] The total share of wealth invested in concerns, startups, and IPOs is calculated, and the percentage difference between the generalized Kelly rule and the NPV strategy is shown in Table 4. In most cases, the difference is well below 0.5%, which shows that the results produced by the two strategies are close to being identical. However, the differences between the two strategies widen massively when the default probability for concerns achieves 5% or more; overall, a generalized Kelly rule investor would invest almost 4% more in concerns than an NPV investor would under such circumstances. This indicates that differences exist between these two types of investors. Since the typical default probability for concerns is 1% or smaller, this difference has no impact on the simulations performed in the rest of this paper, wherein the NPV strategy is used as a proxy for the generalized Kelly rule because it can be calculated much faster than the latter can.

The previous strategy is called the theoretical NPV strategy since the parameters are assumed to be known. But in reality, the true parameters of the dividend model are unknown. Therefore, these parameters are estimated on the basis of past (simulated) data in order to compare this model with other models. D^C and D^S can be directly observed from the data, thereby making probability estimation quite simple. For example, p_{CD} can be estimated by dividing the number of defaults of concerns by the sum of the active concern years of all the concerns plus the number of defaults. This estimator is also a maximum likelihood estimator (MLE). With the estimated parameters, the competing strategy $\lambda^2_{k,t}\left(\omega^t\right)$ can be calculated in the same way as $\lambda^1_{k,t}\left(\omega^t\right)$, and it is called the empirical NPV strategy.

The next step is to find some interesting alternative strategies. Hens et al. (2002) and Hens and Schenk-Hoppé (2004) applied a simple strategy; they used average relative dividends as a proxy for λ^*. In the case of i.i.d. dividends, this is the Kelly rule. Therefore this strategy is

$$\lambda^3_{k,t}\left(\omega^t\right) = \frac{c_3}{\tau+1}\sum_{i=0}^{\tau}\frac{D^k_{t-i}\left(\omega^{t-i}\right)}{\sum_j D^j_{t-i}\left(\omega^{t-i}\right)} := \frac{c_3}{\tau+1}\sum_{i=0}^{\tau} d^k_{t-i}(\omega^{t-i}).$$

The number of periods over which averaging has been conducted is represented by τ and the factor c_3 is chosen such that $\sum_{k=0}^{K}\lambda^3_{k,t}\left(\omega^t\right) = 1$. This constant is needed because the environment of the existing companies differs

[6]The parameters D^S, $p_{SS} = 1 - p_{SC} - p_{SD}$ and $p_{CC} = 1 - p_{CD}$ are not varied either because they are normed to one or given by the other parameters.

Panel A: Varying D^C

D^C		2	10	50	100
IPO:	$\lambda^*_{IPO} - \lambda^1_{IPO}$	0.021%	-0.001%	0.001%	-0.003%
Startup:	$\mathbb{E}\left(n^S\right) \cdot \left(\lambda^*_S - \lambda^1_S\right)$	0.156%	0.102%	0.165%	0.205%
Concern:	$\mathbb{E}\left(n^C\right) \cdot \left(\lambda^*_C - \lambda^1_C\right)$	-0.177%	-0.101%	-0.166%	-0.202%

Panel B: Varying p_{SC}

p_{SC}		10%	5%	2%	1%
IPO:	$\lambda^*_{IPO} - \lambda^1_{IPO}$	0.031%	-0.006%	-0.004%	0.000%
Startup:	$\mathbb{E}\left(n^S\right) \cdot \left(\lambda^*_S - \lambda^1_S\right)$	0.304%	0.258%	0.198%	0.214%
Concern:	$\mathbb{E}\left(n^C\right) \cdot \left(\lambda^*_C - \lambda^1_C\right)$	-0.336%	-0.252%	-0.194%	-0.215%

Panel C: Varying p_{SD}

p_{SD}		10%	5%	2%	1%
IPO:	$\lambda^*_{IPO} - \lambda^1_{IPO}$	-0.010%	-0.013%	-0.001%	-0.005%
Startup:	$\mathbb{E}\left(n^S\right) \cdot \left(\lambda^*_S - \lambda^1_S\right)$	0.100%	0.105%	0.120%	0.174%
Concern:	$\mathbb{E}\left(n^C\right) \cdot \left(\lambda^*_C - \lambda^1_C\right)$	-0.090%	-0.091%	-0.120%	-0.169%

Panel D: Varying p_{CD}

p_{CD}		10%	5%	2%	1%
IPO:	$\lambda^*_{IPO} - \lambda^1_{IPO}$	-0.144%	-0.120%	-0.019%	-0.009%
Startup:	$\mathbb{E}\left(n^S\right) \cdot \left(\lambda^*_S - \lambda^1_S\right)$	-3.585%	-3.637%	-0.430%	0.137%
Concern:	$\mathbb{E}\left(n^C\right) \cdot \left(\lambda^*_C - \lambda^1_C\right)$	3.728%	3.757%	0.449%	-0.127%

Table 4: Percentage difference between the Kelly rule, λ^*, and the NPV strategy, λ^1, in terms of total investment in IPOs, startups, and concerns. The parameters of the dividend process can be found in Table 3. In each panel, one parameter is varied. The strategies are calculated on the assumption that the number of IPOs, startups, and concerns is equal to their long-run averages of 1, $\mathbb{E}\left(n^S\right)$ and $\mathbb{E}\left(n^C\right)$. The generalized Kelly rule, λ^*, is obtained through 10,000 simulations over 1,000 time periods.

in every time period and it is therefore not a given that the full budget of the agents is used after the averaging. This paper uses this strategy with the τ values of 100, 20, and 0. Considering a long history to estimate the relative dividends is effective in the case of i.i.d. relative dividends. In that case, the strategy converges to λ^*. This is not the case in the selected dividend model, but this strategy is still an important benchmark. The case of $\tau = 0$ is special in that it relies only on the current relative dividends. Therefore, this strategy is the called current relative dividend strategy. Since it relies only on an extremely short history, it may be in a strong position in a setup where not much can be learnt from the past dividend history.

The last strategy diversifies naively; it invests the same amount into all existing assets. In other words,

$$\lambda_{k,t}^{5}\left(\omega^{t}\right) = \frac{1}{\text{Total number of active companies in } t}.$$

This strategy may appear somewhat unsophisticated, but DeMiguel et al. (2009) have showed that it performs astonishingly well on real data.

Obviously, many more strategies are possible. However, the generalized Kelly rule performs best in simulations within the i.i.d. and stationary setting, whereas mean-variance, adaptive, and even more sophisticated strategies have no chance of surviving (see Hens et al. (2002), Hens and Schenk-Hoppé (2004), and Tupak (2009)). Therefore, it makes sense to examine mainly those strategies on the basis of relative dividends, such as the generalized Kelly rule. To ensure that this inference holds, Section 6.3 compares the NPV strategy with a large number of fixed-mix strategies and confirms that the NPV strategy is not only a surviving strategy but, perhaps, also a locally evolutionary stable strategy.

6 Simulations

The main purpose of this section is to simulate the wealth dynamic of the competing investment strategies described above within the dividend process detailed herein. This will be done using equation (10). First, a simple example demonstrates the errors that can be produced by choosing an inadequate strategy by wrongly assuming a stationary dividend process. Second, simulations show that the NPV strategy is indeed able to take over a large part of the market, and finally, some robustness checks are performed on the NPV strategy.

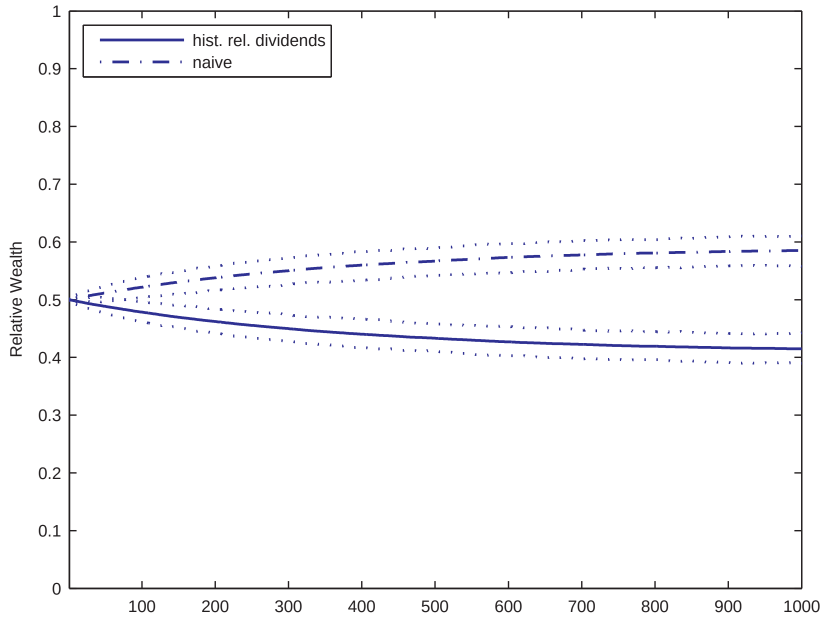


Figure 4: Relative wealth mean and 95% confidence intervals out of 1,000 simulations over 1,000 time periods. The market comprises two strategies: the historical relative dividend strategy averaged over the previous 20 time periods and the naive strategy. All strategies begin with equal wealth.

6.1 An illustrative example

With i.i.d. relative dividends, the Kelly rule λ^* is equal to the average past relative dividends, λ^3. This subsection shows that λ^3 is unable to dominate the market under the dividend model of this paper with nonstationary dividends and finitely lived assets, and it fails against the naive strategy of investing the same amount into each asset, λ^5. In contrast to our paper, most of the literature assumes i.i.d. dividends with infinitely lived assets for their simulations (see, e.g., Hens et al. (2002), Hens and Schenk-Hoppé (2004), or Tupak (2009)); this questions the relevance of these results. The dividend process is parameterized according to Table 3. To emphasize the results of this section, p_{SC} was set at 0.15 (as a consequence, $p_{SS} = 0.827$). This example establishes that a naive strategy that invests an equal share of wealth in every company can accumulate more wealth than a strategy based on the average relative dividends of the previous 20 periods (see Figure 4). The two strategies begin with equal wealth, and after 1,000 periods averaged out over 1,000 simulations, the naive $1/n$-strategy claims 58.5% of the total wealth, whereas the historical relative dividend strategy accounts for 41.5%.[7] [8] This result follows from the high probability that a startup will become a concern, which the relative dividend investor, who invests according to the past average relative dividends, neither knows nor takes into account because the event is observed once in the life of a firm. In contrast, the $1/n$-strategy increases its wealth share by investing more funds in startups than the relative dividend strategy does. Therefore, both strategies will survive in the long run. However, parameterizations calibrated on CRSP data show that the success of the naïve investor in the real world falls far short of the outcome achieved in this example. The simulated parameters in Table 3 put the real-world probability of a startup becoming a concern at a mere 0.7% (not 15%). This parameter results in an average wealth share of just 6.2% for the $1/n$-strategy after 1,000 periods. The simulations wherein $p_{SC} = 0.15$ are an example of how an optimal strategy may completely fail if a wrong dividend process is assumed. Therefore, it is extremely important for evolutionary simulations to assume a correctly specified dividend process.

6.2 Is the NPV strategy able to take over the market?

The main purpose of this section is to demonstrate that an NPV strategy will finally take over almost the entire market. To achieve this objective, the parameters given in Table 3 were used to simulate changes in the wealth shares of the NPV, current relative dividend, historical relative dividend, and naive strategies. This involved estimating the mean and 95% confidence intervals of the relative wealth of the competing strategies on the basis of 1,000 simulations (see Figure 5). The results confirm that the theoretical NPV strategy outperforms all the other strategies to a striking extent. However, the strategies converge very slowly compared to those used by Hens et al. (2002) and Hens and Schenk-Hoppé (2004), who use i.i.d. dividend processes. In particular, the current relative dividend strategy loses wealth shares so slowly that after 1,000 periods, it still commands a larger market share than the NPV strategy does. Given that the model was calibrated so that one period equates to one year, this implies that evolutionary convergence can require an extremely long time horizon, especially if the competing strategies are not

[7] Increasing the number of time periods to 5,000 (the default number of time periods in the rest of the paper) does not have a significant impact on the result.

[8] The simulations of the whole paper were also done with 100 simulations. The impact on average wealth, the main variable of interest, is minor. The only visible difference was that the confidence bands became smoother with 1,000 simulations. An even higher number of simulations is therefore unlikely to alter the results.

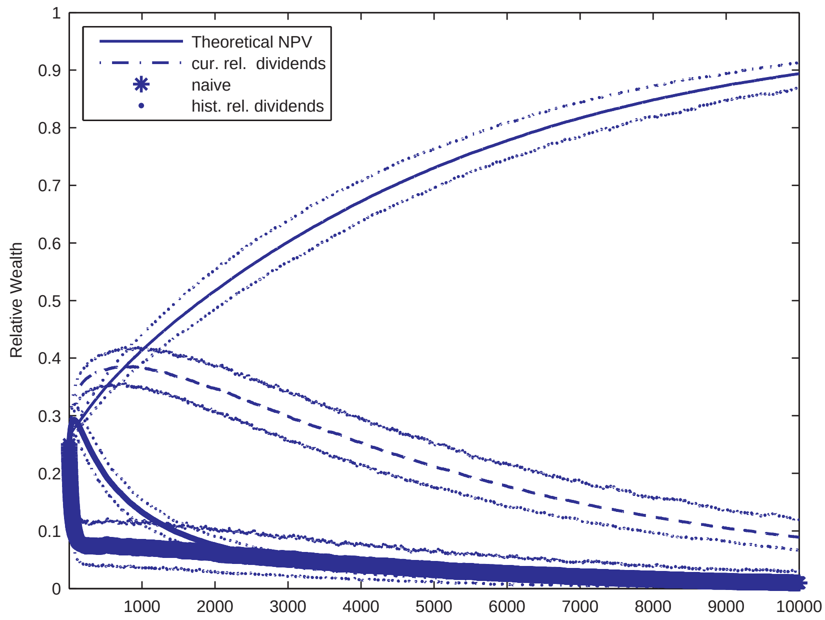


Figure 5: Relative wealth mean and 95% confidence intervals out of 1,000 simulations over 10,000 time periods. The market comprises four strategies: the theoretical NPV strategy, the current relative dividends strategy, the historical relative dividend strategy averaged over the previous 20 time periods, and the naive strategy. All strategies begin with equal wealth.

very dissimilar from the optimal one.

As mentioned earlier, a locally evolutionary stable strategy prevents invading strategies from earning higher returns when it has almost all the wealth in a market. Is the theoretical NPV strategy a locally evolutionary stable strategy? This question is easily answered via simulation. The process involves assuming that the theoretical NPV strategy begins with 97% of total wealth and that the three other strategies each start off with 1% of total wealth. On the basis of this assumption, 5,000 time periods are then simulated 1,000 times in order to determine whether the theoretical NPV strategy is able to retain its wealth share. The results of this procedure reveal that, after 5,000 periods, the theoretical NPV strategy owns an average of 97.7% of total wealth, with a standard deviation of 0.9%. In contrast, the current relative dividend strategy accrues an average of 1.3% of total wealth with a standard deviation of 0.6%. The simulated distribution of the latters increase in wealth shows that it is not statistically significant at the 5% level. In other words, strategies with small total wealth shares are not able to wrest market share away from the theoretical NPV strategy. Therefore, the theoretical NPV strategy is evolutionary stable, at least against the chosen alternative strategies.

The wealth shares of the NPV strategy should also be determined using dividend parameters that are estimated from simulated dividend data, that is, the empirical NPV strategy. To this end, the parameters of the dividend process must be estimated over a sufficient number of time periods to ensure that the parameter values are precise enough. For example, if the parameters for calculating the empirical NPV strategy are determined on the basis of the previous 20 periods at every point of time, then the empirical strategy would be vanquished by the current relative dividend strategy. Starting with 97% of total wealth, the wealth share of the empirical NPV strategy would fall to an average of 57.0% of total wealth after 5,000 periods simulated 1,000 times each. Conversely, the wealth share of the current relative dividends strategy would expand from 1% to 28.5% of total wealth, and the historical relative dividends strategy and the naive strategy would gain 5.3% and 7.2% of additional wealth share, respectively. Figure 6 depicts the results of simulations using parameters obtained on the basis of the previous 100 periods. Both the empirical NPV and historical relative dividend strategies are now determined over the previous 100 periods so that the learning horizon remains consistent between them. In the same setup, the empirical NPV strategy acquires 98.0% of total wealth after 5,000 periods. All other strategies lose in wealth share aside from the relative dividend strategy, whose wealth share grows fractionally from 1% to 1.1%. In contrast to the results generated

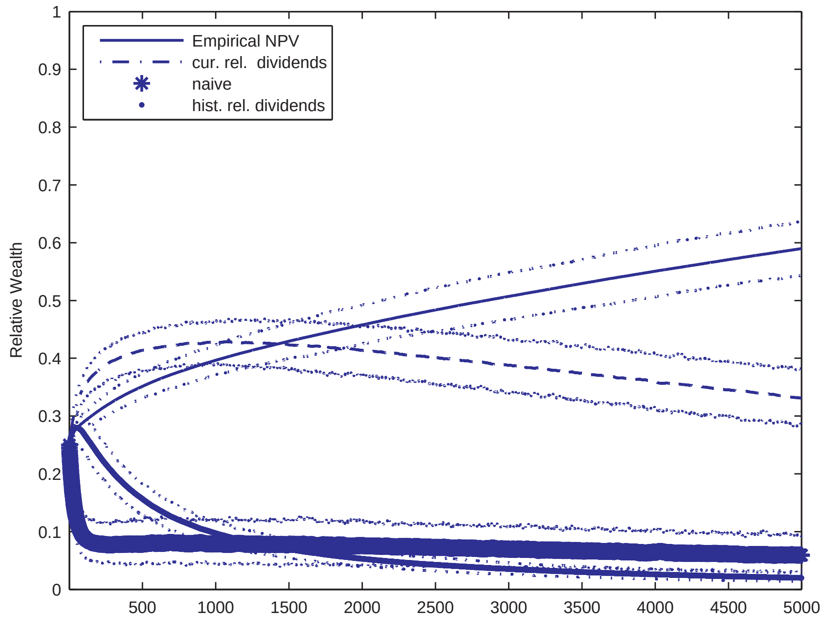
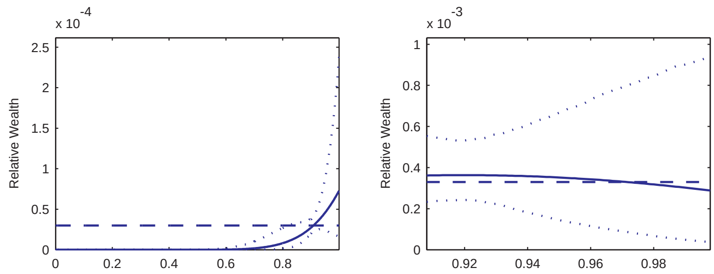


Figure 6: The relative wealth mean and 95% confidence intervals of 1,000 simulations of 5,000 time periods. The market comprises four strategies: the empirical NPV strategy, the current relative dividends strategy, the historical relative dividend strategy averaged over 100 periods, and the naïve strategy. All strategies start with equal wealth.

by parameters estimated from the previous 20 periods, these results show that the empirical NPV strategy easily conquers the other strategies when parameters calculated from the previous 100 periods are used. However, the empirical NPV strategy converges at a slower pace than it does in the simulations for the theoretical NPV strategy. This leads to the conclusion that, by definition, an imprecise estimation of the correct parameters affects the performance of the empirical NPV strategy, in some cases so much that the empirical NPV strategy has no chance to survive. This concurs with Tupak (2009), who finds that other strategies can perform better than λ^* if the latter must be learnt from the data. For infinitely lived assets and strategies that depend only on the state of the world, Theorem 2 of Amir et al. (2009a) implies that the optimal strategy learnt from the data must not survive against the optimal strategy that knows the true model parameters, that is, other strategies may triumph over the estimated optimal strategy. The simulations by DeMiguel et al. (2009) showed that thousands of monthly observations are required before an optimal mean-variance strategy featuring asset returns that possess a multivariate normal distribution can overcome the naive $1/n$ strategy. This is mainly because the estimated average returns of the strategies contain a high level of error. Thus, the implementation of the theoretical optimal strategy may remain a challenge because of the errors in the estimation of the dividend process.

6.3 Robustness Checks

Obviously, the set of strategies in the market can greatly influence the outcome; therefore, the NPV strategy should be tested against as many other strategies as practicable in order to confirm the findings presented above. This was accomplished by further running the theoretical NPV strategy against a wide range of fixed-mix strategies. The fraction of funds invested in IPOs was assumed constant at 0.2% of total investment, which is the rounded average value of the NPV strategy if the number of companies is given by the long-run average defined in equation (1). Total investment in concerns was varied between 0% and 99.8% of total investment and was calculated using quantities that differed by 0.1% from each other. This investment was equally divided between all concerns and the rest of the investment was equally divided between all startups. This resulted in the creation of 998 different fixed-mix strategies to compete with the NPV strategies in the market. The theoretical NPV strategy started with 97% of total wealth with the rest di-



(a) Fixed-mix strategies with an investment share of between 0% and 99.8% in concerns after 5,000 periods

(b) Fixed-mix strategies with an investment share of between 90.8% and 99.8% in concerns after 5,000 periods

Figure 7: The full line indicates the wealth share of fixed-mix strategies investing a constant share of wealth into concerns as averaged over 1,000 simulations. The dotted lines represent the 95% confidence interval of the strategies. The percentage of investment in concerns is shown on the x-axis. The percentage of investment in IPOs holds steady at 0.2% of the total investment and the remainder is invested in startups. The dashed line represents the initial wealth share.

vided equally among the fixed-mix strategies.[9] Figure 6.3 shows the average wealth share of the various fixed-mix strategies after 5,000 periods simulated 1,000 times each. The dashed line represents the average wealth share of these strategies in the first period. Fixed-mix strategies that invested less than 90.8% into concerns lose in average market share, while other fixed-mix strategies gain. Overall, the NPV strategy is able to increase its wealth share and ends up with 99.3% of total wealth. Naturally, the gains of the fixed-mix strategies that invested heavily in concerns are obtained at the expense of the fixed-mix strategies that invested limited amounts in concerns. To compare the successful fixed-mix strategies with the NPV strategies, all fixed-mix strategies that had gained in average wealth shares (i.e., those that invested 90.8% or more of their total wealth in concerns) were then matched against the NPV strategy. As before, all strategies involved were simulated 1,000 times per period over 5,000 periods. After 5,000 periods, the NPV strategy amasses a wealth share of 96.9%. The 95% confidence band rises from 93.9% to 98.6%, indicating that the initial wealth of the strategy does not differ statistically from the final wealth. In contrast, none of the fixed-mix strate-

[9] This percentage was chosen to make the setup comparable with that delineated in Section 6.2. With an initial wealth of 90% of the theoretical NPV strategies, the results are comparable and the theoretical NPV strategy gains massively in wealth shares.

gies with a 95% confidence interval are able to gain a statistically significant proportion of wealth shares and their wins or losses more or less amount to zero (see Figure 7(b)). Overall, no fixed-mix strategy is able to push the NPV strategy out of the market. On the other hand, the NPV strategy is also unable to push the fixed-mix strategies completely out of the market (although the wealth share of the latter is small). This evinces that the NPV strategy is a very close approximation to the real dominant strategy but is not itself the dominant strategy (given that such a strategy exists at all).

The results obtained in the previous section may differ according to variations in model parameters. Therefore, I ran additional tests in order to evaluate the main hypothesis that the NPV strategy is locally evolutionary stable compared with the current relative dividend, historical relative dividend, and naive investment strategies. This was done by estimating the aforementioned strategies with several different sets of parameterizations and examining the stability of the results thereby obtained. The NPV strategy was simulated with parameters calculated from the previous 100 periods and an initial wealth share of 97% and allocated the other strategies 1% each of wealth share. The results show increases in the wealth share of the empirical NPV strategy averaged out over 1,000 simulations over 5,000 time periods (see Table 5) and shows that this strategy can increase its relative weight under different parameterizations. This suggests that the NPV strategy can survive the evolutionary timeline with a large share of wealth and may therefore be at least close to a locally evolutionary stable strategy.

7 Conclusion

This paper demonstrates that large dividend jumps and the creation, growth, and default of companies are very important aspects of the dividend process. The idealized model presented herein shows that these factors have considerable influence on the performance of different investment strategies and signifies the inadequacy of considering only the time series of any given company in determining the percentage of wealth that should be invested into that company. Rather, these findings suggest that comparable companies should be studied in order to determine the optimal portfolio weight of companies. This is a new idea that complicates many aspects of evolutionary finance, including the estimation of an appropriate dividend model, and constitutes an important drawback: even very primitive strategies can outperform the most elaborate ones if a huge amount of data is required to estimate them accurately. Nevertheless, the NPV strategy, a close substitute of the gen-

	Emp. NPV	Current rel. div.	Historical rel. div.	Naive investor
Initial relative wealth	0.970	0.010	0.010	0.010
Benchmark	0.980	0.011	0.000	0.008
	(0.002)	(0.001)	(0.000)	(0.002)
$\lambda_0 = 10\%$	0.982	0.012	0.000	0.007
	(0.002)	(0.001)	(0.000)	(0.002)
$d_C = 10$	0.975	0.013	0.002	0.010
	(0.002)	(0.001)	(0.000)	(0.002)
$d_C = 100$	0.984	0.009	0.000	0.007
	(0.002)	(0.001)	(0.000)	(0.002)
$n_{new} = 5$	0.978	0.012	0.002	0.008
	(0.001)	(0.000)	(0.000)	(0.001)
$p_{CC} = 98\%$	0.977	0.017	0.000	0.006
$p_{CD} = 2\%$	(0.008)	(0.005)	(0.000)	(0.006)
$p_{SC} = 2.0\%$	0.972	0.013	0.006	0.009
$p_{SS} = 95.7\%$	(0.002)	(0.001)	(0.001)	(0.001)
$p_{SD} = 5.0\%$	0.973	0.015	0.000	0.011
$p_{SS} = 92.3\%$	(0.004)	(0.002)	(0.000)	(0.003)

Table 5: The average relative wealth shares of the four main strategies after 1,000 simulations of 5,000 periods. The figures in parentheses express the standard deviations of the given percentages. The market comprises four investors: the empirical NPV investor (learning over 100 periods), the current relative dividends investor, the historical relative dividend investor averaging over 100 time periods, and the naive investor. Eight models were estimated: the benchmark model, which was calculated according to the parameterizations in Table 3, and seven other models wherein one parameter has been different compared to the benchmark model. The varied parameter and its new value can be found in the leftmost column.

eralized Kelly rule, dominates within this setup, although it requires long time periods to approximate a 100% wealth share. Alternatively, this result could also indicate that the strategy that is able to achieve the most precise calculations of the fundamental value of a firms dividends will be the one to survive or even dominate the market in the long run.

Simulation studies, such as this one, inherently face one major issue: it is never possible to test the whole range of possible parameters. Therefore, even with the extensive robustness checks carried out within the paper, there is no guarantee that the results found can be generalized for all cases. Furthermore, the Kelly strategy does not generalize to the chosen setup and must be approximated by the NPV strategy. Therefore, the present paper provides only a rough approximation of an evolutionary stable strategy.

This study generates three interesting directions for future research. First, theoretical results pertaining to nonstationary dividends and finitely lived firms would ascertain whether the results provided by simulations in this paper are generally applicable. Second, future work could perform simulations on the basis of alternative stochastic dividend processes in order to investigate the impact of such processes on the surviving strategy. Third, this paper has shown that the learning period may wield a crucial influence on strategy performance. Therefore, future work could attempt to determine how the dividend process should be learned optimally, such that the optimal strategy based on those results is able to take over the market.

References

Rabah Amir, Igor V. Evstigneev, Thorsten Hens, and Klaus Reiner Schenk-Hoppé. Market Selection and Survival of Investment Strategies. *Journal of Mathematical Economics*, 41(1-2):105–122, February 2005.

Rabah Amir, Igor V. Evstigneev, Thorsten Hens, and Le Xu. Evolutionary Finance and Dynamic Games. Working Paper 581, NCCR "Financial Valuation and Risk Management", September 2009a.

Rabah Amir, Igor V. Evstigneev, and Klaus Reiner Schenk-Hoppé. Asset Market Games of Survival. Working Paper 505, NCCR "Financial Valuation and Risk Management", September 2009b.

Larry Blume and David Easley. If You're So Smart, Why Aren't You Rich? Belief Selection in Complete and Incomplete Markets. *Econometrica*, 74 (4):929–966, November 2006.

Lawrence Blume and David Easley. Evolution and Market Behavior. *Journal of Economic Theory*, 58(1):9–40, July 1992.

John Y. Campbell and Robert J. Shiller. Cointegration and Tests of Present Value Models. *Journal of Political Economy*, 95(5):1062–88, October 1987.

John Y. Campbell and Robert J. Shiller. The Dividend-Price Ratio and Expectations of Future Dividends and Discount Factors. *Review of Financial Studies*, 1(3):195–228, 1988.

Sudheer Chava and Robert A. Jarrow. Bankruptcy Prediction with Industry Effects. *Review of Finance*, 8(4):537–569, 2004.

Hui Chen. Macroeconomic Conditions and the Puzzles of Credit Spreads and Capital Structure. Journal of Finance, forthcoming, 2010.

Harry DeAngelo and Linda DeAngelo. The Irrelevance of the MM Dividend Irrelevance Theorem. *Journal of Financial Economics*, 79(2):293–315, February 2006.

Harry DeAngelo, Linda DeAngelo, and Rene M. Stulz. Dividend Policy and the Earned / Contributed Capital Mix: A Test of the Life-Cycle Theory. *Journal of Financial Economics*, 81(2):227–254, August 2006.

David N. DeJong and Charles H. Whiteman. The Temporal Stability of Dividends and Stock Prices: Evidence from the Likelihood Function. *American Economic Review*, 81(3):600–617, June 1991.

Victor DeMiguel, Lorenzo Garlappi, and Raman Uppal. Optimal Versus Naive Diversification: How Inefficient is the 1-N Portfolio Strategy? *Review of Financial Studies*, 22(5):1915–1953, May 2009.

Rajeev Dhawan. Firm Size and Productivity Differential: Theory and Evidence from a Panel of US Firms. *Journal of Economic Behavior & Organization*, 44(3):269–293, March 2001.

David S. Evans. Tests of Alternative Theories of Firm Growth. *Journal of Political Economy*, 95(4):657–674, August 1987a.

David S. Evans. The Relationship between Firm Growth, Size, and Age: Estimates for 100 Manufacturing Industries. *Journal of Industrial Economics*, 35(4):567–581, June 1987b.

Igor Evstigneev, Thorsten Hens, and Klaus Reiner Schenk-Hoppé. Market Selection of Financial Trading Strategies: Global Stability. *Mathematical Finance*, 12(4):329–339, October 2002.

Igor Evstigneev, Thorsten Hens, and Klaus Reiner Schenk-Hoppé. Evolutionary Stable Stock Markets. *Economic Theory*, 27(2):449–468, January 2006.

Igor V. Evstigneev, Thorsten Hens, and Klaus Reiner Schenk-Hoppé. Globally Evolutionarily Stable Portfolio Rules. *Journal of Economic Theory*, 140(1):197 – 228, 2008.

Eugene F. Fama. The Behavior of Stock-Market Prices. *Journal of Business*, 38(1):34–105, January 1965.

Eugene F. Fama and Kenneth R. French. Disappearing Dividends: Changing Firm Characteristics or Lower Propensity to Pay? *Journal of Financial Economics*, 60(1):3–43, April 2001.

Milton Friedman. *Essays in Positive Economics*. Chicago: University of Chicago Press, 1953.

Gustavo Grullon, Roni Michaely, and Bhaskaran Swaminathan. Are Dividend Changes a Sign of Firm Maturity? *Journal of Business*, 75(3):387–424, July 2002.

Bronwyn H. Hall. The Relationship between Firm Size and Firm Growth in the U.S. Manufacturing Sector. *Journal of Industrial Economics*, 35(4): 583–606, June 1987.

Richard D. F. Harris and Elias Tzavalis. Testing for Unit Roots in Dynamic Panels in the Presence of a Deterministic Trend: Re-Examining the Unit Root Hypothesis for Real Stock Prices and Dividends. *Econometric Reviews*, 23(2):149–166, January 2004.

Thorsten Hens and Klaus Reiner Schenk-Hoppé. Survival of the Fittest on Wall Street. Working Paper 151, NCCR Finrisk, February 2004.

Thorsten Hens and Klaus Reiner Schenk-Hoppé. Evolutionary Stability of Portfolio Rules in Incomplete Markets. *Journal of Mathematical Economics*, 41(1-2):43–66, February 2005.

Thorsten Hens, Klaus Reiner Schenk-Hoppé, and Martin Stalder. An Application of Evolutionary Finance to Firms Listed in the Swiss Market Index. *Swiss Journal of Economics and Statistics*, 138(4):465–487, 2002.

William J. Hurley and Lewis D. Johnson. A Realistic Dividend Valuation Model. *Financial Analysts Journal*, 50(3):50–54, July/August 1994.

Roger G. Ibbotson and Jeffrey F. Jaffe. "Hot Issue" Markets. *Journal of Finance*, 30(4):1027–1042, September 1975.

John L. Kelly. A New Interpretation of Information Rate. *Bell System Technical Journal*, 35:917–926, July 1956.

Allan W. Kleidon. Variance Bounds Tests and Stock Price Valuation Models. *Journal of Political Economy*, 94(5):953–1001, October 1986.

J. Bradford De Long, Andrei Shleifer, Lawrence H. Summers, and Robert J. Waldmann. Positive Feedback Investment Strategies and Destabilizing Rational Speculation. *Journal of Finance*, 45(2):374–97, June 1990.

Dennis C. Mueller. A Life Cycle Theory of the Firm. *Journal of Industrial Economics*, 20(3):199–219, July 1972.

Lubos Pástor and Pietro Veronesi. Rational IPO Waves. *Journal of Finance*, 60(4):1713–1757, August 2005.

Paul A. Samuelson. Evolution and Market Behavior. *Journal of Banking & Finance*, 3(4):305–307, December 1979.

Alvaro Sandroni. Do Markets Favor Agents Able to Make Accurate Predicitions? *Econometrica*, 68(6):1303–1342, November 2000.

Robert J. Shiller. The Use of Volatility Measures in Assessing Market Efficiency. *Journal of Finance*, 36(2):291–304, May 1981.

Andreas Tupak. *Essays in Numerical Evolutionary Finance*. PhD thesis, University of Zurich, 2009.

Maria Vassalou and Yuhang Xing. Default Risk in Equity Returns. *Journal of Finance*, 59(2):831–868, April 2004.